

# INTERPOLATION AND APPROXIMATION OF POLYNOMIALS IN FINITE FIELDS OVER A SHORT INTERVAL FROM NOISY VALUES

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ABSTRACT. Motivated by a recently introduced HIMMO key distribution scheme, we consider a modification of the *noisy polynomial interpolation problem* of recovering an unknown polynomial  $f(X) \in \mathbb{Z}[X]$  from approximate values of the residues of  $f(t)$  modulo a prime  $p$  at polynomially many points  $t$  taken from a short interval.

## 1. INTRODUCTION

1.1. **Motivation.** Here, we consider the following problem:

Given a prime  $p$ , recover an unknown polynomial  $f$  over a finite field  $\mathbb{F}_p$  of  $p$  elements from several approximations to the values  $f(t)$ , computed at several points  $t$ .

Several problems of this type are related to the so-called *hidden number problem* introduced by Boneh and Venkatesan [2, 3], and have already been studied intensively due to their cryptographic relevance, see the survey [26].

Usually the evaluation points  $t$  are chosen from the whole field  $\mathbb{F}_p$ . However, here, motivated by the links and possible applications to the recently introduced HIMMO key distribution scheme [8], we concentrate on a very different case, which has never been discussed in the literature. Namely, in the settings relevant to HIMMO, the values of  $t$  are taken from a short interval rather than from the whole field  $\mathbb{F}_p$ . This case requires a careful adaptation of existing algorithms of [25, 27] and also establishing new number theoretic results about the frequency of small residues of polynomial values evaluated at a small argument, which are based on some ideas from [4, 5].

We remark the polynomial recovery problem as studied in this paper arises in a collusion attack on a single node in the HIMMO system.

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Collusion attacks on the complete HIMMO system lead to recovery problems involving several polynomials reduced modulo several distinct unknown integers, which we believe to be much more difficult.

**1.2. Polynomial noisy interpolation and approximation problems.** For a prime  $p$  we denote by  $\mathbb{F}_p$  the field of  $p$  elements. We identify the elements of  $\mathbb{F}_p$  with the set  $\{0, \dots, p-1\}$ , so we can talk about their bits (for example most and least significant bits) and also approximations.

We consider the *noisy polynomial interpolation problem* of finding an unknown polynomial

$$f(X) = \sum_{j=0}^n a_j X^j \in \mathbb{F}_p[X],$$

of degree  $n$  from approximations to values of  $f(t)$  (treated as elements of the set  $\{0, \dots, p-1\}$ ) at polynomially many points  $t \in \mathbb{F}_p$  selected uniformly at random.

More precisely, for integers  $s$  and  $m \geq 1$  we denote by  $|s|_m$  the distance between  $s$  and the closest multiple of  $m$ , that is,

$$|s|_m = \min_{k \in \mathbb{Z}} |s - km|.$$

Then, these approximations can, for example, be given as integers  $u_t$  with

$$(1) \quad |u_t - f(t)|_p \leq \Delta$$

for some “precision”  $\Delta$  for values of  $t$  that are chosen uniformly at random from a certain set  $\mathcal{T}$  of “samples”.

In cryptographic applications, the approximations are usually given by strings of  $s$  most or least significant bits of  $f(t)$ , where, if necessary, some leading zeros are added to make sure that  $f(t)$  is represented by  $r$  bits, where  $r$  is the bit length of  $p$ .

It is clear that giving the most significant bits is essentially equivalent to giving approximations of the type (1) with an appropriate  $\Delta$ . We now observe that giving the least significant bits can be recast as a question of this type as well. Indeed, assume we are given some  $s$ -bit integer  $v_t$  so that

$$f(t) = 2^s w_t + v_t$$

for some (unknown) integer  $w_t$ . We now have  $0 \leq w_t < 2^{r-s}$  so, if  $\lambda$  is the multiplicative inverse of  $2^s$  in  $\mathbb{F}_p$  then, setting  $u_t = \lambda v_t + 2^{r-s-1}$ , we obtain an inequality of the type (1) with the polynomial  $\lambda f(X)$  instead of  $f(X)$  and  $\Delta = 2^{r-s-1}$ .

So from now on we concentrate on the case where the approximations to  $f(t)$  are given in the form of (1).

The case of linear polynomials corresponds to the hidden number problem introduced by Boneh and Venkatesan [2, 3]. The case of general polynomials, including sparse polynomials of very large degree, has been studied in [25, 27] (see also [26] for a survey of several other problems of similar types). We note that if the polynomial  $f$  belongs to some family  $\mathcal{F}$  of polynomials (such as polynomials of degree  $d$  or sparse polynomials with  $d$  monomials), it is crucial for the algorithms of [25, 27] to have some uniformity results for every non-zero polynomial of  $\mathcal{F}$  on the values of  $t$  chosen uniformly at random from  $\mathcal{T}$ . We also note that a multiplicative analogue of this problem has been studied in [9].

We recall that in the settings of all previous works, typically the set  $\mathcal{T}$  is the whole field  $\mathbb{F}_p$ . In turn, this enables the use of such a powerful number theoretic technique as the method of exponential sums and thus the use of the Weil bound (see [17, Theorem 5.38]) for dense polynomials and the bound of Cochrane, Pinner and Rosenhouse [6] for sparse polynomials. In particular, in [27] a polynomial time algorithm is designed that works with very large values of  $\Delta$ , namely, up to  $\Delta = p \exp(-c(w \log p)^{1/2})$ , provided that  $f$  is a sparse polynomial of degree  $n \leq p^{1/2} \exp(-c(w \log p)^{1/2})$  and with  $w$  monomials (of known degrees), where  $c > 0$  is some absolute constant. The analysis of the algorithm of [27] is based on the method of exponential sums, the bound of [6] and some ideas related to the Waring problem.

We also note that if  $\mathcal{T}$  is an interval of length at least  $p^{1/n+\varepsilon}$  for some fixed  $\varepsilon > 0$ , where, as before  $n = \deg f$ , then one can use the method of [25, 27] augmented with the bounds of exponential sums obtained via the method of Vinogradov, see the striking results of Wooley [28, 29, 30]. However, here we are interested in shorter intervals of the form

$$\mathcal{T} = [-h, h],$$

which appear naturally in the construction of [8]. Thus instead of the method of exponential sums, we use some ideas from [4, 5].

We remark that for the noisy interpolation to succeed, we always assume that some low order coefficients  $a_0, \dots, a_{k-1}$  of  $f$  are known, where  $k$  is such that  $h^k > \Delta^{1+\varepsilon}$  for some fixed  $\varepsilon > 0$ . Clearly such a condition is necessary as if  $h^k \leq \Delta$  then the approximations of the type (1) are not likely to be sufficient to distinguish between

$$f_1(X) = \sum_{j=k}^n a_j X^j \quad \text{and} \quad f_2(X) = \sum_{j=k}^n a_j X^j + X^{k-1}$$

from approximations (1) at random  $t \in [-h, h]$ . Furthermore, it is clear that without loss of generality we can always assume that

$$a_0 = \dots = a_{k-1} = 0.$$

Note that in the case when the test set  $\mathcal{T}$  is the whole field  $\mathbb{F}_p$  we only need to request that the constant coefficient  $a_0$  of  $f$  is known (which is also, always assumed to be zero, see [2, 3, 25, 27]).

The above example, which shows the limits of *interpolation* of  $f$  from the information given by (1), motivates the following question of *approximation* to  $f$ , which is also more relevant to attacking HIMMO. Namely, instead of finding a polynomial  $f$  we ask whether we can find a polynomial  $\tilde{f}$ , such that  $f(t)$  and  $\tilde{f}(t)$  are close to each other for all  $t \in [-h, h]$  so the approximations (1) do not allow to distinguish between  $f$  and  $\tilde{f}$ . In the terminology of [8] both  $f$  and  $\tilde{f}$  lead to the same keys and thus the attacker can use  $\tilde{f}$  instead of  $f$ .

Finally, we note that the algorithmic problems considered in this paper have led us to some new problems which are of intrinsic number theoretic interest.

**1.3. Approach and structure.** Generally, our approach follows that of [25, 27] which in turn is based on the idea of Boneh and Venkatesan [2, 3]. Thus lattice algorithms, namely the algorithm for the closest vector problem, see Section 3.1, and a link between this problem and polynomial approximations, see Section 3.2, play a crucial role in our approach.

However, the analysis of our algorithm requires very different tools compared with those used in [25, 27]. Namely we need to establish some results about the frequency of small polynomial values at small arguments, which we derive in Section 2.1, closely following some ideas from [4, 5].

It also clear that there is a natural limit for such estimates on the frequency of small values as any polynomial with small coefficients takes small values at small arguments (it is certainly easy to quantify the notion of “small” in this statement). In fact our bound of Lemma 1 is nontrivial up to exactly this limit.

Rather unexpectedly, in Section 2.2 we show that there are several other types of polynomials which satisfy this property: that is, they take small values at small arguments even if the coefficients are quite large (certainly this may only happen beyond the range of the bound of Lemma 1 which applies to all polynomials).

We call such polynomials *exceptional*. Then, after recalling in Section 2.3 some result from analytic number theory, in Section 2.4 we give

some results describing the structure of the coefficients of exceptional polynomials. Note the presence of exceptional polynomials makes noisy polynomial interpolation impossible for a wide class of instances (which is actually good news for the security of HIMMO). So it is important to understand the structure and the frequency of exceptional polynomials. The results of Section 2.4 provide some partial progress toward this goal, yet many important questions are still widely open.

Our main results are presented in Sections 4.1 and 6. In Section 4.1 we treat the noisy polynomial interpolation problem, where we actually want to recover the hidden polynomial  $f$ . However, from the cryptographic point of view it is enough to recover another polynomial  $g$  which for all or most of small arguments takes values close to those taken by  $f$ . Here the existence of exceptional polynomials becomes of primal importance. We use the results of Section 2.4 to obtain an upper bound on the number of such polynomials  $g$ .

Furthermore, in Sections 5.1 and 5.2 we also give two interesting explicit constructions of such exceptional polynomials, which we illustrate by some concrete numerical examples.

In Section 6 we treat the approximate recovery problem and give a formula that predicts whether an approximate recovery is likely to be successful, depending on the number of randomly chosen observation points. We compare its predictions with numerical experiments for parameter values suitable for HIMMO.

**1.4. Notation.** Throughout the paper, the implied constants in the symbols ‘ $O$ ’, ‘ $\ll$ ’ and ‘ $\gg$ ’ may occasionally, where obvious, depend on the degrees of the polynomials involved and on the real parameter  $\varepsilon$  and are absolute, otherwise. We recall that the notations  $U = O(V)$ ,  $U \ll V$  and  $V \gg U$  are all equivalent to the assertion that the inequality  $|U| \leq c|V|$  holds for some constant  $c > 0$ .

The letter  $p$  always denotes a prime number, while the letters  $h$ ,  $k$ ,  $\ell$ ,  $m$  and  $n$  (in lower and upper cases) always denote positive integer numbers.

## 2. DISTRIBUTION OF VALUES OF POLYNOMIALS

**2.1. Polynomial values in a given interval.** For a polynomial

$$F(X) \in \mathbb{F}_p[X]$$

and two intervals

$$\mathcal{I} = \{u + 1, \dots, u + H\} \quad \text{and} \quad \mathcal{J} = \{v + 1, \dots, v + K\}$$

of  $H$  and  $K$  consecutive integers, respectively, we denote by  $N_F(\mathcal{I}, \mathcal{J})$  the number of values  $t \in \mathcal{I}$  for which  $F(t) \in \mathcal{J}$  (where the elements of the intervals  $\mathcal{I}$  and  $\mathcal{J}$  are embedded into  $\mathbb{F}_p$  under the natural reduction modulo  $p$ ).

For two intervals  $\mathcal{I}$  and  $\mathcal{J}$  of the same length, various bounds on  $N_F(\mathcal{I}, \mathcal{J})$  are given in [4, 5]. It is easy to see that the argument of the proof of [5, Theorem 1] allows us to estimate  $N_F(\mathcal{I}, \mathcal{J})$  for intervals of  $\mathcal{I}$  and  $\mathcal{J}$  of different lengths as well.

To present this result, for positive integers  $k, \ell$  and  $H$ , we denote by  $J_{k,\ell}(H)$  the number of solutions to the system of equations

$$x_1^\nu + \dots + x_k^\nu = x_{k+1}^\nu + \dots + x_{2k}^\nu, \quad \nu = 1, \dots, \ell,$$

with

$$1 \leq x_1, \dots, x_{2k} \leq H.$$

Next, we define  $\kappa(\ell)$  as the smallest integer  $\kappa$  such that for  $k \geq \kappa$  there exists a constant  $C(k, \ell)$  depending only on  $k$  and  $\ell$  and such that

$$J_{k,\ell}(H) \leq C(k, \ell) H^{2k - \ell(\ell+1)/2 + o(1)}$$

holds as  $H \rightarrow \infty$ . Presently, the strongest available upper bound

$$\kappa(\ell) \leq \ell^2 - \ell + 1$$

is due to Wooley [30], see also [29, Theorem 1.1] and [28, Theorem 1.1]; note that, in particular, these bounds also imply the existence of  $\kappa(\ell)$  which is a very nontrivial fact.

It is now easy to see that the proof of [5, Theorem 1] can be generalised to lead to the following bound.

**Lemma 1.** *Let  $F \in \mathbb{F}_p[X]$  be a polynomial of degree  $\ell$ . Then for intervals  $\mathcal{I} = \{u + 1, \dots, u + H\}$  and  $\mathcal{J} = \{v + 1, \dots, v + K\}$  with  $1 \leq H, K < p$  we have*

$$N_F(\mathcal{I}, \mathcal{J}) \leq H^{1+o(1)} \left( (K/p)^{1/2\kappa(\ell)} + (K/H^\ell)^{1/2\kappa(\ell)} \right),$$

as  $H \rightarrow \infty$ .

Note that Kerr [15, Theorem 3.1] gives a more general form of Lemma 1 that applies to multivariate polynomials and also to congruences modulo a composite number.

We also note, that several other results from [4, 5] can be extended from the case  $H = K$  to the general case as well, but this does not affect our main result.

Clearly the bound of Lemma 1 is nontrivial provided that  $K \leq p^{1-\varepsilon}$  and also  $H^\ell > K^{1+\varepsilon}$  for a fixed  $\varepsilon > 0$  and a sufficiently large  $H$ . It is easy to see that this conditions cannot be substantially improved as it is clear that if  $K = p$  then  $N_F(\mathcal{I}, \mathcal{J}) = H$  for any polynomial

$F \in \mathbb{F}_p[X]$ . Also, if  $K = H^\ell$  then  $N_F(\mathcal{I}, \mathcal{J}) = H$  for  $\mathcal{I} = \{1, \dots, H\}$ ,  $\mathcal{J} = \{1, \dots, K\}$  and  $F(X) = X^\ell$ . The above example may suggest that unless the coefficients of  $F$  are small, the value of  $N_F(\mathcal{I}, \mathcal{J})$  can be nontrivially estimated even for  $K > H^\ell$ . We now show that this is false and in fact there are many other polynomials, which we call *exceptional*, which have the same property.

**2.2. Exceptional polynomials.** Let us fix integers  $s \geq 2$ ,  $0 \leq r_i < s^i$ ,  $i = 0, \dots, \ell$  and  $K > \ell H^\ell$ . We now consider a polynomial

$$F(X) = \sum_{i=0}^{\ell} A_i X^i \in \mathbb{F}_p[X],$$

where

$$(2) \quad A_i \in \left[ \frac{r_i}{s^i} p, \frac{r_i}{s^i} p + \frac{K}{\ell H^i} \right], \quad i = 0, \dots, \ell.$$

In particular  $A_i s^i \equiv B_i \pmod{p}$  for some integers

$$B_i \in \left[ 0, \frac{K s^i}{\ell H^i} \right], \quad i = 0, \dots, \ell.$$

Then for  $t = su$  with an integer  $u \in [1, H/s]$  we have

$$F(t) = F(su) = \sum_{i=0}^{\ell} A_i (su)^i \equiv \sum_{i=0}^{\ell} B_i u^i \pmod{p}.$$

Hence  $F(t) \in [0, K]$  for such values of  $t$ , which implies the inequality  $N_F(\mathcal{I}, \mathcal{J}) \geq H/s - 1$  for the above example. Taking  $s$  to be small, we see that  $F$  has quite large coefficients, however no nontrivial bound on  $N_F(\mathcal{I}, \mathcal{J})$  is possible for  $K > \ell H^\ell$ .

In fact, in Sections 5.1 and 5.2 we give examples of polynomials with large coefficients that remain small for all small values of the arguments, that is, polynomials for which  $N_F(\mathcal{I}, \mathcal{J}) = H$ , for some rather short intervals  $\mathcal{J}$  compared with the length of  $\mathcal{I}$ .

In the above examples, the ratios of the coefficients and  $p$  are chosen to be very close to rational numbers with small numerators and denominators. Alternatively, this can be written as the condition that the coefficients of  $F$  satisfy

$$A_i v \equiv u_i \pmod{p}$$

for some small integers  $v, u_i$ ,  $i = 0, \dots, \ell$ .

Below we present several results that demonstrate that this property indeed captures the class of exceptional polynomials  $F$  with large values of  $N_F(\mathcal{I}, \mathcal{J})$  adequately, at least in the qualitative sense.

First we need to recall some number theoretic results.

**2.3. Some preparations.** The following result is well-known and can be found, for example, in [20, Chapter 1, Theorem 1] (which is a more precise form of the celebrated Erdős–Turán inequality).

**Lemma 2.** *Let  $\gamma_1, \dots, \gamma_H$  be a sequence of  $H$  points of the unit interval  $[0, 1]$ . Then for any integer  $R \geq 1$ , and an interval  $[\alpha, \beta] \subseteq [0, 1]$ , we have*

$$\begin{aligned} & \#\{t = 1, \dots, H : \gamma_t \in [\alpha, \beta]\} - H(\beta - \alpha) \\ & \ll \frac{H}{R} + \sum_{r=1}^R \left( \frac{1}{R} + \min\{\beta - \alpha, 1/r\} \right) \left| \sum_{t=1}^H \exp(2\pi i r \gamma_t) \right|. \end{aligned}$$

To use Lemma 2 we also need an estimate on exponential sums with polynomials, which is essentially due to Weyl, see [14, Proposition 8.2].

Let  $|\xi|_1 = \min\{|\xi - k| : k \in \mathbb{Z}\}$  denote the distance between a real  $\xi$  and the closest integer (which can be considered as a modulo 1 version of the notation  $|z|_p$ ).

**Lemma 3.** *Let  $\psi(X) \in \mathbb{R}[X]$  be a polynomial of degree  $\ell \geq 2$  with the leading coefficient  $\vartheta \neq 0$ . Then*

$$\begin{aligned} & \left| \sum_{t=1}^H \exp(2\pi i \psi(t)) \right| \\ & \ll H^{1-\ell/2^{\ell-1}} \left( \sum_{-H < t_1, \dots, t_{\ell-1} < H} \min \left\{ H, \frac{1}{|\vartheta \ell! t_1 \cdots t_{\ell-1}|_1} \right\} \right)^{1/2^{\ell-1}}. \end{aligned}$$

Finally, we need the following version of several similar and well known inequalities, see, for example, [14, Section 8.2].

**Lemma 4.** *For any integers  $w, v, H$  and  $Z$  with*

$$\gcd(v, w) = 1, \quad H \geq 1, \quad 2Z > v \geq 1$$

*and a real  $\alpha$  with*

$$\left| \alpha - \frac{w}{v} \right| < \frac{1}{2Zv}$$

*we have*

$$\sum_{z=1}^Z \min \left\{ H, \frac{1}{|\alpha z|_1} \right\} \ll HZv^{-1} + Z \log v.$$



*Proof.* First we estimate the contribution from  $z \equiv 0 \pmod{v}$  trivially as  $HZ/v$ . For the remaining  $z$  we note that

$$|\alpha z|_1 \geq \left| \frac{wz}{v} \right|_1 - \frac{z}{2Zv} \geq \frac{1}{2} \left| \frac{wz}{v} \right|_1$$

as

$$\left| \frac{wz}{v} \right|_1 \geq \frac{1}{v} \quad \text{and} \quad \frac{z}{2Zv} \leq \frac{1}{2v}.$$

Therefore, the contribution from  $z \not\equiv 0 \pmod{v}$  can be estimated as

$$\begin{aligned} \sum_{\substack{z=1 \\ z \not\equiv 0 \pmod{v}}}^Z \min \left\{ H, \frac{1}{|\alpha z|_1} \right\} &\ll \sum_{\substack{z=1 \\ z \not\equiv 0 \pmod{v}}}^Z \left| \frac{wz}{v} \right|_1^{-1} \\ &\leq (Z/v + 1) \sum_{z=1}^{v-1} \left| \frac{wz}{v} \right|_1^{-1}. \end{aligned}$$

Since  $\gcd(v, w) = 1$  and  $v \leq 2Z$  this simplifies as

$$\begin{aligned} \sum_{\substack{z=1 \\ z \not\equiv 0 \pmod{v}}}^Z \min \left\{ H, \frac{1}{|\alpha z|_1} \right\} &\ll Zv^{-1} \sum_{z=1}^{v-1} \left| \frac{z}{v} \right|_1^{-1} \\ &\ll Z \sum_{1 \leq z \leq v/2} \frac{1}{z} \ll Z \log v, \end{aligned}$$

which concludes the proof.  $\square$

**2.4. Coefficients of exceptional polynomials.** Our first result uses in an essential way the argument of the proof of [5, Theorem 3],

**Lemma 5.** *Let*

$$F(X) = \sum_{i=0}^{\ell} A_i X^i \in \mathbb{F}_p[X]$$

*be a polynomial of degree  $\ell$ . Assume that for some  $\rho \leq 1$  and intervals  $\mathcal{I} = \{1, \dots, H\}$  and  $\mathcal{J} = \{1, \dots, K\}$  with  $1 \leq H, K < p$  we have*

$$N_F(\mathcal{I}, \mathcal{J}) \geq \max\{(2\ell + 1), \rho H\}.$$

*Then*

$$A_i v \equiv u_i \pmod{p}$$

*for some integers*

$$1 \leq v \ll \rho^{-\ell(\ell+1)/2} \quad \text{and} \quad u_i \ll \rho^{-\ell(\ell-1)/2} K H^{\ell-i}, \quad i = 0, \dots, \ell.$$

*Proof.* As we have mentioned, we follow the proof of [5, Theorem 3].

Let  $N = N_F(\mathcal{I}, \mathcal{J})$ . Since  $N \geq 2(\ell + 1)$ , there exist  $\ell + 1$  pairs  $(x_1, y_1), \dots, (x_{\ell+1}, y_{\ell+1})$  such that  $x_1, \dots, x_{\ell+1}$  lie in an interval  $\tilde{\mathcal{I}}$  of length  $2(\ell + 1)H/N$  and  $y_1, \dots, y_{\ell+1} \in \mathcal{J}$ .

Now, we consider the system of congruences

$$\begin{cases} A_0 + A_1 x_1 + \dots + A_{\ell} x_1^{\ell} & \equiv y_1 \pmod{p}, \\ \dots & \dots \\ A_0 + A_1 x_{\ell+1} + \dots + A_{\ell} x_{\ell+1}^{\ell} & \equiv y_{\ell+1} \pmod{p}. \end{cases}$$

The determinant  $v$  of this system is the determinant of a Vandermonde matrix:

$$v = \det \begin{pmatrix} 1 & x_1 & \dots & x_1^{\ell} \\ \dots & \dots & \dots & \dots \\ 1 & x_{\ell+1} & \dots & x_{\ell+1}^{\ell} \end{pmatrix} = \prod_{1 \leq i < j \leq d+1} (x_j - x_i).$$

Note that  $v \not\equiv 0 \pmod{p}$ . Thus, we have that

$$(3) \quad v A_i \equiv u_i \pmod{p},$$

where

$$(4) \quad \begin{aligned} u_i &= \det \begin{pmatrix} 1 & \dots & x_1^{i-1} & y_1 & x_1^{i+1} & \dots & x_1^{\ell} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \dots & x_{\ell+1}^{i-1} & y_{\ell+1} & x_{\ell+1}^{i+1} & \dots & x_{\ell+1}^{\ell} \end{pmatrix} \\ &= \sum_{j=1}^{\ell+1} (-1)^{i+j} y_j V_{ij} \end{aligned}$$

and  $V_{ij}$  is the determinant of the matrix obtained from the Vandermonde matrix after removing the  $j$ -th row and the  $i$ -th column,  $i = 0, \dots, \ell$ ,  $j = 1, \dots, \ell + 1$ .

It is easy to see that for each  $i = 0, \dots, \ell$ , the determinant  $V_{ij}$  is a polynomial in  $\ell$  variables  $x_{\nu}$ ,  $1 \leq \nu \leq \ell + 1$ ,  $\nu \neq j$ , of degree  $\ell(\ell + 1)/2 - i$ , which vanishes when  $x_r = x_s$  for distinct  $r$  and  $s$ . Thus

$$V_{ij} = W_i(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{\ell+1}) \prod_{\substack{1 \leq r < s \leq \ell+1 \\ r, s \neq j}} (x_s - x_r),$$

where  $W_i$  is a polynomial (that does not depend on  $F$  or  $p$ ) of degree  $\ell - i$ . Therefore, we have

$$V_{ij} \ll (H/N)^{\ell(\ell-1)/2} H^{\ell-i} \leq \rho^{-\ell(\ell-1)/2} H^{\ell-i}.$$

This estimate and (4) together imply the bound

$$(5) \quad u_i \ll \rho^{-\ell(\ell-1)/2} K H^{\ell-i}, \quad i = 0, \dots, \ell.$$

On the other hand, it is clear that

$$(6) \quad v \ll (H/N)^{\ell(\ell+1)/2} \leq \rho^{-\ell(\ell+1)/2}.$$

Then, comparing (3) with (5) and (6) we conclude the proof.  $\square$

We note that if  $s$  and  $\rho$  are fixed the expressions given by the example (2) and those of Lemma 5 differ by a factor  $H^\ell$  and it is certainly natural to try to reduce this gap.

We now obtain yet another estimate on  $N_F(\mathcal{I}, \mathcal{J})$  that depends only on the behaviour of the leading coefficient.

We recall the definition of  $\kappa(\ell)$  used in Lemma 1. Since for  $H^\ell > K$  we can use the bound of Lemma 1, we assume that

$$(7) \quad 2\ell!H^{\ell-1} \leq K.$$

Then the argument of the proof of [4, Theorem 5] leads to the following result:

**Lemma 6.** *Let*

$$F(X) = \sum_{i=0}^{\ell} A_i X^i \in \mathbb{F}_p[X]$$

*be a polynomial of degree  $\ell$ . Assume that for some  $\rho \leq 1$  and intervals  $\mathcal{I} = \{1, \dots, H\}$  and  $\mathcal{J} = \{1, \dots, K\}$  with  $1 \leq H, K < p$  satisfying (7) we have*

$$N_F(\mathcal{I}, \mathcal{J}) \geq \rho H.$$

*Then either*

$$\rho \ll \min\{K/p, H^{-1/2^{\ell-1}} p^{o(1)}\},$$

*or*

$$A_\ell v \equiv u \pmod{p}$$

*for some integer*

$$1 \leq v \leq \rho^{-2^{\ell-1}} p^{o(1)} \quad \text{and} \quad u \ll H^{-\ell+1} K.$$

*Proof.* Let  $N = N_F(\mathcal{I}, \mathcal{J})$ . We apply Lemma 2 to the sequence of fractional parts  $\gamma_n = \{F(t)/p\}$ ,  $t \in \mathcal{I}$ , with

$$\alpha = 1/p, \quad \beta = K/p, \quad R = \lfloor p/K \rfloor.$$

Without loss of generality, we can assume that  $K < p/2$  as otherwise the bound is trivial. Then we have  $R \geq 1$  and thus

$$\frac{1}{R} + \min\{\beta - \alpha, 1/r\} \ll \frac{K}{p}$$

for  $r = 1, \dots, R$ , we derive

$$N \ll \frac{HK}{p} + \frac{K}{p} \sum_{r=1}^R \left| \sum_{t=1}^H \exp(2\pi i r F(t)/p) \right|.$$

Therefore, by Lemma 3, we have

$$N \ll \frac{HK}{p} + \frac{H^{1-\ell/2^{\ell-1}} K}{p} \times \sum_{r=1}^R \left( \sum_{-H < t_1, \dots, t_{\ell-1} < H} \min \left\{ H, \left| \frac{A_\ell}{p} \ell! r t_1 \cdots t_{\ell-1} \right|_1^{-1} \right\} \right)^{1/2^{\ell-1}}.$$

Now, separating from the sum over  $t_1, \dots, t_{\ell-1}$  the terms with  $t_1 \cdots t_{\ell-1} = 0$ , (giving a total contribution  $O(H^{\ell-1})$ ), we obtain

$$(8) \quad \begin{aligned} N &\ll \frac{HK}{p} + \frac{H^{1-\ell/2^{\ell-1}} K}{p} R(H^{\ell-1})^{1/2^{\ell-1}} + \frac{H^{1-\ell/2^{\ell-1}} K}{p} W \\ &\ll \frac{HK}{p} + H^{1-1/2^{\ell-1}} + \frac{H^{1-\ell/2^{\ell-1}} K}{p} W, \end{aligned}$$

where

$$W = \sum_{r=1}^R \left( \sum_{0 < |t_1|, \dots, |t_{\ell-1}| < H} \min \left\{ H, \left| \frac{A_\ell}{p} \ell! r t_1 \cdots t_{\ell-1} \right|_1^{-1} \right\} \right)^{1/2^{\ell-1}}.$$

Hence, recalling the choice of  $R$ , we derive

$$(9) \quad N \ll HKp + H^{1-1/2^{\ell-1}} + \frac{H^{1-\ell/2^{\ell-1}} K}{p} W.$$

The Hölder inequality implies the bound

$$W^{2^{\ell-1}} \ll R^{2^{\ell-1}-1} \sum_{r=1}^R \sum_{0 < |t_1|, \dots, |t_{\ell-1}| < H} \min \left\{ H, \left| \frac{A_\ell}{p} \ell! r t_1 \cdots t_{\ell-1} \right|_1^{-1} \right\}.$$

Collecting together the terms with the same value of

$$z = \ell! r t_1 \cdots t_{\ell-1}$$

and using the well-known bound on the divisor function, we conclude that

$$W^{2^{\ell-1}} \ll R^{2^{\ell-1}-1} p^{o(1)} \sum_{0 < |z| < \ell! H^{\ell-1} R} \min \left\{ H, \left| \frac{A_\ell}{p} z \right|_1^{-1} \right\}.$$

Let us set  $Z = \ell! H^{\ell-1} R$ . Note that our assumption (7) implies that.

$$2Z < p.$$

By the Dirichlet approximation theorem we can write

$$\left| \frac{A_\ell}{p} - \frac{w}{v} \right| \leq \frac{1}{2Zv}$$

for some integers  $w$  and  $1 \leq v < 2Z$ . Note that this immediately implies that  $|A_\ell v - pw| \leq p/2Z$ . Hence

$$A_\ell v \equiv u \pmod{p}$$

for some integer  $u$  with

$$|u| \leq \frac{p}{2Z} \ll H^{-\ell+1} K.$$

It now remains to estimate  $\rho$  or  $v$ .

We apply Lemma 4 with  $Z = \ell! H^{\ell-1} R$ , and we derive:

$$\begin{aligned} W^{2^{\ell-1}} &\ll R^{2^{\ell-1}-1} p^{o(1)} (H^\ell R v^{-1} + H^{\ell-1} R) \\ &= R^{2^{\ell-1}} p^{o(1)} (H^\ell v^{-1} + H^{\ell-1}), \end{aligned}$$

which after the substitution in (8) implies

$$N \ll \frac{HK}{p} + H^{1-1/2^{\ell-1}} + \frac{HK R v^{-1/2^{\ell-1}}}{p^{1+o(1)}} + \frac{H^{1-\ell/2^{\ell-1}} K}{p^{1+o(1)}} R (H^{\ell-1})^{1/2^{\ell-1}}.$$

Noting that the last term, up to the factor  $p^{o(1)}$  coincides with the second term, we now obtain

$$N \ll \frac{HK}{p} + H^{1-1/2^{\ell-1}} p^{o(1)} + H v^{-1/2^{\ell-1}} p^{o(1)}.$$

Hence we either have

$$\rho H \ll \frac{HK}{p} + H^{1-1/2^{\ell-1}} p^{o(1)}$$

or

$$\rho H \ll H v^{-1/2^{\ell-1}} p^{o(1)},$$

which concludes the proof.  $\square$

Note that the dependence on  $\rho$  in the bound on  $v$  of Lemma 6 is worse than that of Lemma 5. However, since we are mostly interested in large values of  $\rho$ , for example, the extreme case  $\rho = 1$  is of special interest, Lemma 6 limits the possible values of  $A_\ell$  as  $O(H^{-\ell+1}K)$ , which is stricter than the bound  $O(K)$  that follows from the estimates of Lemma 5 on  $v$  and  $u_\ell$ .

### 3. LATTICES AND POLYNOMIALS

**3.1. Background on lattices.** As in [2, 3], and then in [25, 27], our results rely on some lattice algorithms. We therefore review some relevant results and definitions, we refer to [7, 10, 11] for more details and the general theory.

Let  $\{\mathbf{b}_1, \dots, \mathbf{b}_s\}$  be a set of  $s$  linearly independent vectors in  $\mathbb{R}^s$ . The set of vectors

$$L = \{\mathbf{z} : \mathbf{z} = \sum_{i=1}^s c_i \mathbf{b}_i, \quad c_1, \dots, c_s \in \mathbb{Z}\}$$

is called an  $s$ -dimensional full rank lattice.

The set  $\{\mathbf{b}_1, \dots, \mathbf{b}_s\}$  is called a *basis* of  $L$ .

The volume of the parallelepiped defined by the vectors  $\mathbf{b}_1, \dots, \mathbf{b}_s$  is called *the volume* of the lattice and denoted by  $\text{Vol}(L)$ . Typically, lattice problems are easier when the Euclidean norms of all basis vectors are close to  $\text{Vol}(L)^{1/s}$ .

One of the most fundamental problems in this area is the *closest vector problem*, CVP: given a basis of a lattice  $L$  in  $\mathbb{R}^s$  and a target vector  $\mathbf{u} \in \mathbb{R}^s$ , find a lattice vector  $\mathbf{v} \in L$  which minimizes the Euclidean norm  $\|\mathbf{u} - \mathbf{v}\|$  among all lattice vectors. It is well known that CVP is **NP**-hard when the dimension  $s \rightarrow \infty$  (see [21, 22, 23, 24] for references).

However, its approximate version admits a deterministic polynomial time algorithm which goes back to the lattice basis reduction algorithm of Lenstra, Lenstra and Lovász [16], see also [21, 22, 23, 24] for possible improvements and further references.

We remark that all lattices that appear in this paper are of fixed dimension, so instead of using the above approximate CVP algorithms, we can simply use the following result of Micciancio and Voulgaris [19].

Let  $\|\mathbf{z}\|$  denote the standard Euclidean norm in  $\mathbb{R}^s$ .

**Lemma 7.** *For any fixed  $s$ , there exists a deterministic algorithm which, for given a lattice  $L$  generated by an integer vectors*

$$\mathbf{b}_1, \dots, \mathbf{b}_s \in \mathbb{Z}^s$$

and a vector  $\mathbf{r} = (r_1, \dots, r_s) \in \mathbb{Z}^s$ , in time polynomial in

$$\sum_{i=1}^s (\log \|b_i\| + \log |r_i| + 1)$$

finds a lattice vector  $\mathbf{v} = (v_1, \dots, v_s) \in L$  satisfying

$$\|\mathbf{v} - \mathbf{r}\| = \min \{\|\mathbf{z} - \mathbf{r}\| : \mathbf{z} \in L\}.$$

**3.2. Lattices and polynomial approximations.** For  $t_1, \dots, t_d \in \mathbb{F}_p$ , integer  $n > k \geq 1$ , we set  $m = n + 1 - k$  and we denote by  $\mathcal{L}_{k,n,p}(t_1, \dots, t_d)$  the  $d + m$ -dimensional lattice generated by the rows of the following  $(d + m) \times (d + m)$ -matrix

$$(10) \quad \begin{pmatrix} p & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & p & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots & & \vdots \\ 0 & 0 & \dots & p & 0 & \dots & 0 \\ t_1^k & t_2^k & \dots & t_d^k & 2\Delta/p & \dots & 0 \\ t_1^{k+1} & t_2^{k+1} & \dots & t_d^{k+1} & 0 & \dots & 0 \\ \vdots & & & & \vdots & \ddots & \vdots \\ t_1^n & t_2^n & \dots & t_d^n & 0 & \dots & 2\Delta/p \end{pmatrix}.$$

The following result is a generalization of several previous results of similar flavour, see [2, 25, 27].

**Lemma 8.** *Let  $\delta > 0$  be fixed and let  $p$  be a sufficiently large prime. Assume that some real numbers  $D \geq \Delta \geq 1$  and integer numbers  $h \geq 1$ ,  $k \geq 1$  satisfy*

$$D \leq \min \{p^{1-\delta}, h^k p^{-\delta}\}.$$

*Then there is an integer  $d$ , depending only on  $n$  and  $\delta$ , such that if*

$$f(X) = \sum_{j=k}^n a_j X^j \in \mathbb{F}_p[X]$$

*and  $t_1, \dots, t_d \in [-h, h]$  are chosen uniformly and independently at random, then with probability at least  $1 - 1/p$  the following holds. For any vector  $\mathbf{s} = (s_1, \dots, s_d, 0, \dots, 0)$  with*

$$|f(t_i) - s_i|_p \leq \Delta, \quad i = 1, \dots, d,$$

*all vectors*

$$\mathbf{v} = (v_1, \dots, v_d, v_{d+1}, \dots, v_{d+m}) \in \mathcal{L}_{k,n,p}(t_1, \dots, t_d)$$

*satisfying*

$$(11) \quad \|\mathbf{v} - \mathbf{s}\| \leq D,$$

are of the form

$$\mathbf{v} = (f(t_1), \dots, f(t_d), a_k \Delta/p, \dots, a_n \Delta/p).$$

*Proof.* Let  $\mathcal{P}_{k,f}$  denote the set of  $p^{n+1-k} - 1$  polynomials

$$(12) \quad g(X) = \sum_{j=k}^n b_j X^j \in \mathbb{F}_p[X]$$

with  $g \neq f$ .

We see from Lemma 1 (taking into account that  $\kappa(\ell)$  is a monotonically non-decreasing function of  $\ell$ ) that for any polynomial  $g \in \mathcal{P}_{k,f}$  the probability  $\rho(g)$  that

$$|f(t) - g(t)|_p \leq 2D$$

for  $t \in [-h, h]$  selected uniformly at random is

$$\rho(g) = h^{o(1)} \left( (D/p)^{1/2\kappa(n)} + (D/h^k)^{1/2\kappa(n)} \right) \leq p^{-\eta}$$

provided that  $p$  is large enough, where  $\eta > 0$  depends only on  $\delta$  and  $n$ .

Therefore, for any  $g \in \mathcal{P}_{k,f}$ ,

$$\Pr \left[ \exists i \in [1, d] : |f(t_i) - g(t_i)|_p > 2D \right] = 1 - \rho(g)^d \geq 1 - p^{-d\eta},$$

where the probability is taken over  $t_1, \dots, t_d \in \mathbb{F}_p$  chosen uniformly and independently at random.

Since  $\#\mathcal{P}_{k,f} < p^{n-k+1} \leq p^n$ , we obtain

$$\begin{aligned} \Pr \left[ \forall g \in \mathcal{P}_{k,f}, \exists i \in [1, d] : |f(t_i) - g(t_i)|_p > 2D \right] \\ \geq 1 - p^{n-\eta d} > 1 - p^{-1} \end{aligned}$$

for  $d > (n+1)\eta^{-1}$ , provided that  $p$  is large enough.

We now fix some integers  $t_1, \dots, t_d$  with

$$(13) \quad \min_{g \in \mathcal{P}_{k,f}} \max_{i \in [1, d]} |f(t_i) - g(t_i)|_p > 2D.$$

Let  $\mathbf{v} \in \mathcal{L}_{k,n,p}(t_1, \dots, t_d)$  be a lattice point satisfying (11).

Clearly, for a vector  $\mathbf{v} \in \mathcal{L}_{k,n,p}(t_1, \dots, t_d)$ , there are some integer  $b_k, \dots, b_n, z_1, \dots, z_d$  such that

$$\mathbf{v} = \left( \sum_{j=k}^n b_j t_1^j - z_1 p, \dots, \sum_{j=k}^n b_j t_d^j - z_d p, b_k \Delta/p, \dots, b_n \Delta/p \right).$$



Suppose that  $b_j \not\equiv a_j \pmod{p}$  for some  $j = k, \dots, n$  and let  $g$  be given by (12). In this case we have

$$\begin{aligned} \|\mathbf{v} - \mathbf{s}\| &\geq \max_{i \in [1, d]} |g(t_i) - s_i|_p \\ &\geq \max_{i \in [1, d]} \left( |f(t_i) - g(t_i)|_p - |s_i - f(t_i)|_p \right) > 2D - \Delta \geq D \end{aligned}$$

that contradicts (11).

Now, if  $b_j \equiv a_j \pmod{p}$ ,  $j = k, \dots, n$ , then for all  $i = 1, \dots, d$  we have

$$\left| \sum_{j=k}^n b_j t_i^j - z_i p \right| = \left| \sum_{j=k}^n b_j t_i^j \right|_p = g(t_i),$$

since otherwise there is  $i \in [1, d]$  such that  $|v_i - g(t_i)| \geq p$  and thus  $|v_i - s_i| \geq p - D$ , which contradicts (11) again. Hence  $\mathbf{v}$  is of the desired form.

As we have seen, the condition (13) holds with probability exceeding  $1 - 1/p$  and the result follows.  $\square$

#### 4. MAIN RESULTS

**4.1. Polynomial interpolation problem.** We are ready to prove the main result. We follow the same arguments as in the proof of [2, Theorem 1] that have also been used for several other similar problems, see [25, 26, 27].

**Theorem 9.** *Let  $\varepsilon > 0$  be fixed and let  $p$  be a sufficiently large prime. Assume that a real  $\Delta \geq 1$  and integers  $p/2 > h \geq 1$ ,  $k \geq 1$  satisfy*

$$h^k > \Delta p^\varepsilon \quad \text{and} \quad \Delta < p^{1-\varepsilon}.$$

*There exists an integer  $d$ , depending only on  $n$  and  $\varepsilon$  and a deterministic polynomial time algorithm  $\mathcal{A}$  such that for any polynomial*

$$f(X) = \sum_{j=k}^n a_j X^j \in \mathbb{F}_p[X]$$

*given integers  $t_1, \dots, t_d \in [-h, h]$  and  $s_1, \dots, s_d \in \mathbb{F}_p$  with*

$$|f(t_i) - s_i|_p \leq \Delta, \quad i = 1, \dots, d,$$

*its output satisfies*

$$\Pr_{t_1, \dots, t_d \in [-h, h]} [\mathcal{A}(t_1, \dots, t_d; s_1, \dots, s_d) = (a_k, \dots, a_n)] \geq 1 - 1/p$$

*if  $t_1, \dots, t_d$  are chosen uniformly and independently at random from  $[-h, h]$ .*

*Proof.* We refer to the first  $d$  vectors in the matrix (10) as  $p$ -vectors and we refer to the other  $m = n + 1 - k$  vectors as power-vectors.

Let us consider the vector  $\mathbf{s} = (s_1, \dots, s_d, s_{d+1}, \dots, s_{d+m})$  where

$$s_{d+j} = 0, \quad j = 1, \dots, m.$$

Multiplying the  $j$ th power-vector of the matrix (10) by  $a_j$  and subtracting a certain multiple of the  $j$ th  $p$ -vector,  $j = k, \dots, n$ , we obtain a lattice point

$$\begin{aligned} \mathbf{w}_f &= (w_1, \dots, w_{d+m}) \\ &= (f(t_1), \dots, f(t_d), a_1\Delta/p, \dots, a_m\Delta/p) \in \mathcal{L}_{k,n,p}(t_1, \dots, t_d), \end{aligned}$$

and thus

$$|w_i - s_i|_p \leq \Delta, \quad i = 1, \dots, d + m.$$

Therefore,

$$(14) \quad \|\mathbf{w} - \mathbf{s}\| \leq \sqrt{d + m}\Delta.$$

Now we can use Lemma 7 to find in polynomial time a lattice vector  $\mathbf{v} = (v_1, \dots, v_d, v_{d+1}, \dots, v_{d+m}) \in \mathcal{L}_{k,n,p}(t_1, \dots, t_d)$  such that

$$\|\mathbf{v} - \mathbf{s}\| = \min \{\|\mathbf{z} - \mathbf{s}\| : \mathbf{z} \in \mathcal{L}_{k,n,p}(t_1, \dots, t_d)\} \leq \|\mathbf{w} - \mathbf{s}\|.$$

Hence, we conclude from (14) that

$$\|\mathbf{w} - \mathbf{v}\| \leq 2\sqrt{d + m}\Delta.$$

Applying Lemma 8 with  $D = \sqrt{2(d + m)}\Delta$  and, say  $\delta = \varepsilon/2$  we see that  $\mathbf{v} = \mathbf{w}_f$  with probability at least  $1 - 1/p$ , and therefore the coefficients of  $f$  can be recovered in polynomial time.  $\square$

**4.2. Polynomial approximation problem.** As we have mentioned, it is impossible to recover a “complete” polynomial of degree  $n$  from the approximations (1) for  $t \in [-h, h]$  for a small value of  $h$ . However, it is interesting to estimate the number of possible “false” candidates  $\tilde{f} \in \mathbb{F}_p[X]$ , which for many  $t \in [-h, h]$  take values close to those of  $f$ .

Namely, for a polynomial  $f \in \mathbb{F}_p[X]$ , and real positive parameter  $\rho \leq 1$  we define  $\#\mathcal{M}_f(\rho, h, \Delta)$  as the set of polynomials  $\tilde{f} \in \mathbb{F}_p[X]$  with

$$\left| \tilde{f}(t) - f(t) \right|_p \leq \Delta$$

for at least  $\rho(2h + 1)$  values of  $t \in [-h, h]$ . For example, if for some fixed  $\varepsilon$  we have

$$h^n \geq \Delta p^\varepsilon \quad \text{and} \quad \Delta < p^{1-\varepsilon},$$

the by Lemma 1 we have  $\mathcal{M}_f(\rho, h, \Delta) = \emptyset$  unless  $\rho \leq p^{-\varepsilon/2\kappa(n)+o(1)}$ . So we now consider slightly smaller values of  $h$ .

**Theorem 10.** *Let  $\varepsilon > 0$  be fixed and let  $p$  be a sufficiently large prime. Assume real  $\Delta \geq 1$  and integers  $p/2 > h \geq 1$ ,  $n \geq 1$  satisfy*

$$h^{n-1} \leq \Delta p^{-\varepsilon} \quad \text{and} \quad \Delta < p^{1-\varepsilon},$$

*for some fixed  $\varepsilon > 0$ . Then for  $\rho \geq \max\{\Delta p^{-1}, h^{-1/2^{n-1}}\} p^\varepsilon$  we have*

$$\#\mathcal{M}_f(\rho, h, \Delta) \ll \rho^{-2^{n-1}-n(n^2+1)/2} \Delta^{n+1} h^{(n^2-n+2)/2} p^{o(1)}.$$

*Proof.* For every  $\tilde{f} \in \mathcal{M}_f(\rho, h, \Delta)$ , we apply Lemma 6 to estimate the number of possible values of the leading coefficient of  $f - \tilde{f}$  as  $\rho^{-2^{n-1}} \Delta h^{-n+1} p^{o(1)}$ . Then we also use Lemma 5 to estimate the number of possible values for the other coefficients of  $f - \tilde{f}$  as

$$\begin{aligned} \rho^{-n(n+1)/2} \prod_{i=0}^{n-1} \rho^{-n(n-1)/2} \Delta h^{n-i} &= \rho^{-n(n+1)/2 - n^2(n-1)/2} \Delta^n h^{n(n+1)/2} \\ &= \rho^{-n(n^2+1)/2} \Delta^{n-1} h^{n(n+1)/2} \end{aligned}$$

which concludes the proof.  $\square$

The case of  $\rho = 1$ , when the recovery of  $f$  from the approximations (1) with  $t \in [-h, h]$  is impossible regardless of the number of queries and the complexity of algorithm is certainly of special interest. We denote  $\overline{\mathcal{M}}_f(h, \Delta) = \mathcal{M}_f(1, h, \Delta)$ .

Under the conditions of  $h$  and  $\Delta$  of Theorem 10 we have

$$\overline{\mathcal{M}}_f(h, \Delta) \ll \Delta^n h^{n(n+1)/2} p^{o(1)}.$$

Clearly if the coefficients of  $\tilde{f} \in \mathbb{F}_p[X]$  are sufficiently close to those of  $f$  then  $\tilde{f} \in \overline{\mathcal{M}}_f(h, \Delta)$ . This argument easily leads to the lower bound

$$\overline{\mathcal{M}}_f(h, \Delta) \gg \Delta^{n+1} h^{-n(n+1)/2}.$$

We now show that this is not necessary and in particular the set  $\overline{\mathcal{M}}_f(h, \Delta)$  contains many polynomials that “visually” look very differently from  $f$ .

## 5. CONSTRUCTIONS OF EXCEPTIONAL POLYNOMIALS

**5.1. Flat polynomials.** Let us take any integer  $v$  and non-zero polynomial  $\Psi_i(X) \in \mathbb{Z}[X]$  of degree  $i = 1, \dots, n$  and with coefficients in the range  $[0, v-1]$  such that

$$\Psi_i(t) \equiv 0 \pmod{v}, \quad t = 0, \dots, v-1.$$

For examples if  $v$  is a square-free integer, we can construct such a polynomial via the Chinese Remainder Theorem from polynomials of the form  $\psi(X)(X^r - X) \in \mathbb{F}_r[X]$  for each prime divisor  $r \mid v$ .

Alternatively, given an integer  $v \geq 1$ , for any  $i$  with

$$(15) \quad v \mid i!$$

one can consider take  $\Psi_i(X) = A_i X(X-1) \dots (X-i+1)$  with an integer  $A_i$  that is relatively prime to  $v$ . We note that the function  $S(v)$ , defined as the smallest  $i$  with the property (15) is called the *Smarandache function* and has been extensively studied in the literature, see [13] and references therein.

Now, for any sufficiently small integers  $u_i$  we define the coefficients  $b_i$  from the congruences  $b_i v \equiv u_i \pmod{p}$ ,  $i = 1, \dots, n$ , and consider the polynomial

$$F(X) \equiv \sum_{i=1}^n b_i \Psi_i(X) \in \mathbb{F}_p[X].$$

It is easy to see that the coefficients of  $F(X)$  can be rather large compared to  $p$ . Furthermore, for every  $t \in [-h, h]$  as the fractions  $\Psi_i(t)/v$ ,  $i = 1, \dots, n$ , take integer values of size  $O(h^i)$ , we have

$$F(t) \equiv \sum_{i=1}^n u_i \frac{\Psi_i(t)}{v} \equiv U \pmod{p},$$

where

$$U \ll \sum_{i=1}^n u_i h^i.$$

Varying the parameter  $v$ , the polynomials  $\Psi_i$  and the integers  $u_i$ ,  $i = 1, \dots, n$ , (and also introducing a small non-zero constant coefficient  $b_0$ ) one can get a large family of such polynomials  $F$  that remain “flat” on short intervals. In particular, one can choose  $v$  to be a product of many small primes. This argument can easily be made more precise with explicit constants, however we instead present an illustrative example of such a polynomial.

Let us fix the prime

$$(16) \quad p = 13850178546024150676274172131557249442552 \\ 857086506417208905998552087,$$

We also set  $h = 2^{31}$ ,  $\Delta = \lfloor p/2^{33} \rfloor$  and for  $i = 1, \dots, 5$ , choose integers  $0 < A_i < p$  such that  $A_i i! \equiv 1 \pmod{p}$ :

$$A_1 = 1$$

$$A_2 = 6925089273012075338137086065778624721276428543 \\ 253208604452999276044$$

$$A_3 = 1154181545502012556356181010963104120212738090 \\ 5422014340754998793406$$

$$A_4 = 9810543136767106729027538593186385021808273769 \\ 608712189641748974395$$

$$A_5 = 1962108627353421345805507718637277004361654753 \\ 921742437928349794879$$

and pick numbers  $c_1, \dots, c_5$  randomly such that  $0 < c_i h^i / i! < \Delta$ , for example:

$$c_1 = 728236268016142987379676454561254599761666551820$$

$$c_2 = 118901258278655898193398330486974890011$$

$$c_3 = 80243828316297659193667769559$$

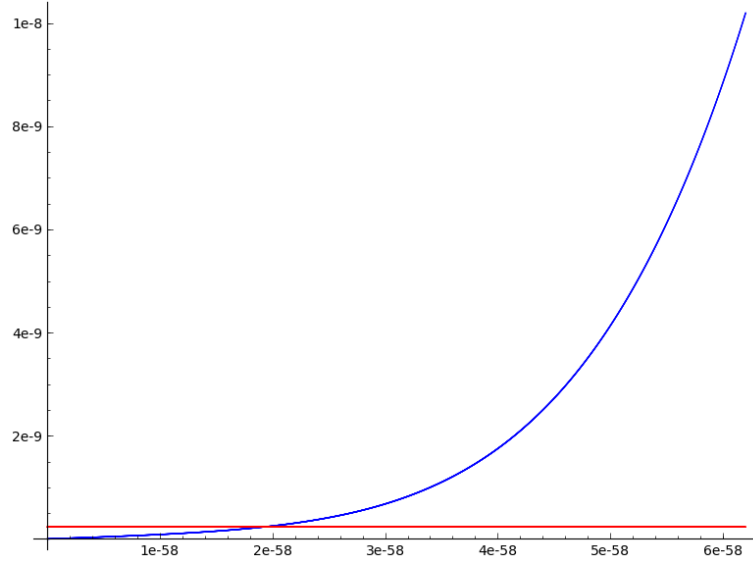
$$c_4 = 177312506479764141124$$

$$c_5 = 210305526612,$$

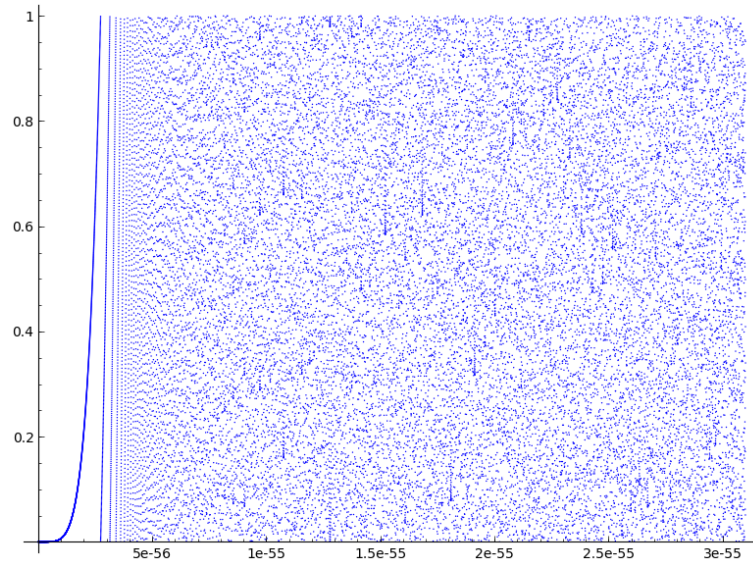
and let

$$F(X) = \sum_{i=1}^5 c_i A_i \prod_{k=0}^{i-1} (X - k).$$

The plot on Figure 1, shows that  $F(x) < \Delta$  (where  $\Delta$  is indicated by the horizontal line) for  $0 \leq x < h$ :

FIGURE 1. Plots of  $F(x)/p$  versus  $x/p$  for  $0 \leq x < 4h$ .

The plot in Figure 2 shows that  $F(x)/p$  stays flat and close to zero for  $x$  close to 0 and, after some transition period, appears to behave completely randomly for  $x \geq 300h$ .

FIGURE 2. Plots of  $F(x)/p$  versus  $x/p$  for  $0 \leq x < 2000h$ .

**5.2. Oscillating polynomials.** A different type of polynomial with large coefficients that is small in an interval can be constructed if  $2^{n+1}\Delta > p$ . Thus this class is interesting only if the degree  $n$  becomes a growing parameter, which is actually the case in the settings of HIMMO.

Let  $f \in \mathbb{Z}[X]$  be polynomial of degree  $n$  that, after reduction modulo  $p$  takes values in the interval  $[-\Delta, \Delta]$  for  $x \in [-h, h]$ . This means that

$$f(X) = d(X) + pc(X)$$

for integer valued functions  $d$  and  $c$ , where for  $x \in [-h, h]$ ,  $d(x) \in [-\Delta, \Delta]$ .

Let  $D$  denote the discrete difference operator, that is, for any function  $A$ ,  $DA(x) = A(x+1) - A(x)$ . We have, since  $f$  is a polynomial of degree  $n$ :

$$0 = D^{n+1}f(x) = D^{n+1}d(x) + pD^{n+1}c(x),$$

so  $D^{n+1}d(x) \in p\mathbb{Z}$ . On the other hand, since  $d(x) \in [-\Delta, \Delta]$  for  $x \in [-h, h]$ , it holds that  $D^{n+1}d(x) \in [-2^{n+1}\Delta, 2^{n+1}\Delta]$  for  $x \in [-h, h-n-1]$ . For  $2^{n+1}\Delta < p$  it follows that  $D^{n+1}d(x) = 0$  for all  $x \in [-h, h-n-1]$ , so that after  $n+1$ -fold integration we have

$$d(x) = \sum_{i=0}^n D^i d(0) \binom{x}{i}.$$

This means that  $d$  and  $c$  are both polynomials of degree at most  $n$  on the interval  $[-h, h]$ .

For  $2^{n+1}\Delta > p$  there are more possibilities. For instance, if

$$D^{n+1}d(x) = p(-1)^x \text{ for } x \in [-h, h-n-1],$$

we obtain, by integrating

$$\begin{aligned} D^n d(x) &= D^n d(0) + \sum_{y=0}^{x-1} D^{n+1}d(y) = D^n d(0) + p \frac{1 - (-1)^x}{2}, \\ D^{n-1}d(x) &= D^{n-1}d(0) + (D^n d(0) + p/2) \binom{x}{1} - p \frac{1 - (-1)^x}{4}, \\ &\vdots \\ d(x) &= d(0) + \sum_{i=1}^n \left( D^i d(0) - \left(-\frac{1}{2}\right)^{n+1-i} p \right) \binom{x}{i} \\ &\quad - (-1)^{n+1} p \frac{1 - (-1)^x}{2^{n+1}}. \end{aligned}$$

Defining

$$c(x) = \sum_{i=0}^n \left(-\frac{1}{2}\right)^{n+1-i} \binom{x}{i} - \left(-\frac{1}{2}\right)^{1+n} (-1)^x,$$

and

$$f(x) = \sum_{i=0}^n D^i d(0) \binom{x}{i},$$

we have a decomposition  $f(x) = d(x) + pc(x)$ , where, for integer  $x$ , these three functions are integer valued. By choosing  $d(0)$  to be close to 0 and, for  $1 \leq i \leq n$ ,  $D^i d(0)$  close to the residues of  $(-1/2)^{1+n-i} p$  modulo  $p$ , we obtain a polynomial  $f$  of degree  $n$  that, after reduction modulo  $p$ , lies in  $[-\Delta, \Delta]$  for arguments in the interval  $[-h, h]$ . For the prime  $p$  given by (16), with  $n = 5$ , and

$$d(0) = 0 \quad \text{and} \quad D^i d(0) \equiv \left\lfloor (-1/2)^{1+n-i} p \right\rfloor \pmod{p}, \quad 1 \leq i \leq 5,$$

where  $\lfloor \xi \rfloor$  denotes the closest integer (and defined arbitrary in case of a tie), we obtain the graph in Figure 3. Note that the left part of the graph in Figure 3 looks like a plot of two curves, while in fact this is one curve, which exhibits a highly oscillatory behaviour.

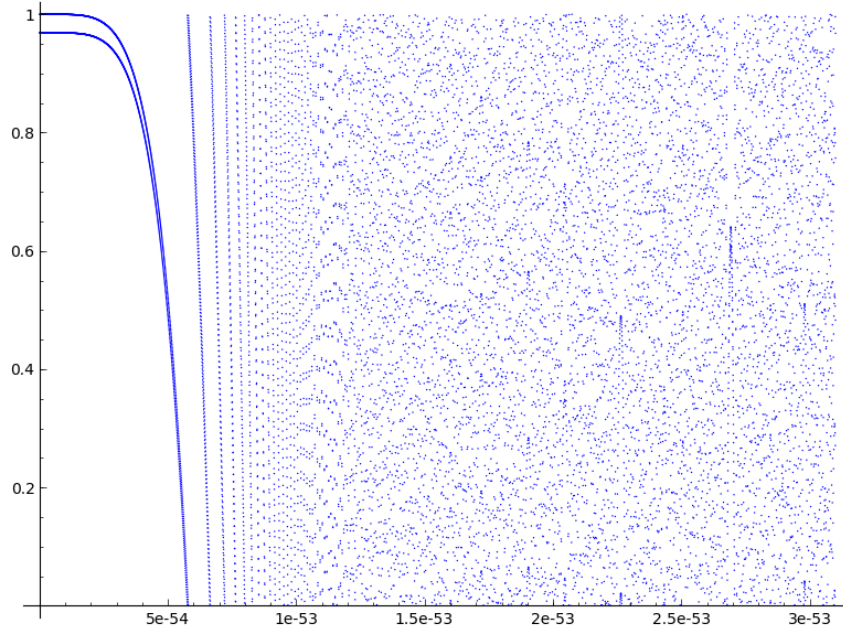


FIGURE 3. Graph of  $f(x)/p$  versus  $x/p$ .



## 6. APPROXIMATE RECOVERY

**6.1. Approach.** Using lattice reduction and rounding techniques, we are able to find a polynomial  $\tilde{f}$  satisfying  $\left|u_t - \tilde{f}(t)\right|_p \leq \Delta$  for  $t \in \{t_1, t_2, \dots, t_d\}$ , the set of observation points. The question is whether  $\tilde{f}(t)$  approximates  $f(t)$  also in all or at least some, non-observed points  $t \in [-h, h] \setminus \{t_1, \dots, t_d\}$ . The lattice used in this technique is spanned by the rows of

$$\begin{pmatrix} p & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & p & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots & & \vdots \\ 0 & 0 & \dots & p & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 & 2\Delta/p & \dots & 0 \\ t_1 & t_2 & \dots & t_d & 0 & \dots & 0 \\ \vdots & & & \vdots & & \ddots & \vdots \\ t_1^n & t_2^n & \dots & t_d^n & 0 & \dots & 2\Delta/p \end{pmatrix},$$

where the first  $d$  columns correspond to the evaluation of the polynomial  $\tilde{f}$  in the points  $t_1, \dots, t_d$ , and the last  $n+1$  columns to its coefficients, scaled such that all coordinates of the wanted lattice point lie in an interval of length  $2\Delta+1$ . In the remainder of this section we focus on the first  $d$  columns, that is, the  $d$ -dimensional lattice  $L$  of which the  $d+n+1$  rows of the matrix

$$\begin{pmatrix} p & 0 & \dots & 0 \\ 0 & p & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & p \\ 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_d \\ \vdots & \vdots & & \vdots \\ t_1^n & t_2^n & \dots & t_d^n \end{pmatrix}$$

are an overcomplete basis.

If  $d \geq n+1$ , a straightforward basis transformation that corresponds to Lagrange interpolation on  $\mathbb{Z}_p$  eliminates  $n+1$  rows and transforms this basis into

$$\begin{pmatrix} pI_{d-n-1} & 0_{d \times (n+1)} \\ M_{(n+1) \times (d-n-1)} & I_{n+1} \end{pmatrix},$$

from which it is clear that  $\text{Vol}(L) = p^{d-n-1}$  (see Section 3.1 for the definition of  $\text{Vol}(L)$ ).

We consider the case that  $2^{n+1}\Delta < p$ , so that, as shown in Section 5.2 all functions  $\tilde{f}$  for which  $\left|\tilde{f}(t) - f(t)\right|_p \leq \Delta$  for all  $t \in [-h, h]$  satisfy

$$\tilde{f}(t) - f(t) = \sum_{i=0}^n A_i \binom{t}{i}$$

for sufficiently small  $A_0, \dots, A_n$ . These differences correspond to the  $n+1$ -dimensional lattice  $L_{\text{approx}}$  in  $\mathbb{Z}^d$  of which the rows of the matrix

$$B_{\text{approx}} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_d \\ \binom{t_1}{2} & \binom{t_2}{2} & \cdots & \binom{t_d}{2} \\ \vdots & \vdots & & \vdots \\ \binom{t_1}{n} & \binom{t_2}{n} & \cdots & \binom{t_d}{n} \end{pmatrix}$$

form a basis. The volume of the lattice corresponding to these approximating polynomials is given by

$$\text{Vol}(L_{\text{approx}}) = \sqrt{\det(B_{\text{approx}} B_{\text{approx}}^T)}$$

The determinant can be expanded as a sum of squares of Vandermonde determinants, yielding

$$\text{Vol}(L_{\text{approx}}) = \prod_{i=0}^n \frac{1}{i!} \times \left( \sum_{\substack{S \subseteq \{1,2,\dots,d\} \\ \#S=n+1}} \prod_{1 \leq k < \ell \leq n+1} (t_{S(k)} - t_{S(\ell)})^2 \right)^{1/2},$$

where the set  $S = \{S(1), S(2), \dots, S(n+1)\}$  runs over all subsets of  $\{1, 2, \dots, d\}$  with  $n+1$  elements.

The lattice technique returns a polynomial  $\tilde{f}$  such that the point

$$\left( \tilde{f}(t_1) - f(t_1), \dots, \tilde{f}(t_d) - f(t_d) \right)$$

lies in the lattice  $L$ . If this point is also a lattice point of  $L_{\text{approx}}$ , then  $\tilde{f}$  approximates  $f$  in all points of  $[-h, h]$ , and the attack succeeds. So, from the attacker point of view, it is good if all lattice vectors of  $L$  that do not lie in  $L_{\text{approx}}$  are long, that is, have the infinity norm greater than  $\Delta$ , since then those points lie outside the hypercube around the target vector.

We define the lattice  $L^\perp$ , obtained by orthogonally projecting all points of  $L$  onto the hyperplane through the origin that is orthogonal

to the basis vectors of  $L_{\text{approx}}$ . Obviously it holds that

$$\text{Vol}(L^\perp) = \frac{\text{Vol}(L)}{\text{Vol}(L_{\text{approx}})}.$$

If  $L^\perp$  does not have short vectors, its volume is large and the attack succeeds. If its volume is small,  $L^\perp$  has short vectors and, depending on the number of independent short vectors and their lengths, the attack is likely to fail.

This leads us to define the function

$$S(n, h, p, \Delta, d) = \log \left( \frac{1}{d\Delta} \left( \frac{\text{Vol}(L)}{\sqrt{\mathbb{E}_h[\text{Vol}(L_{\text{approx}})^2]}} \right)^{1/(d-n-1)} \right).$$

It compares an estimate of the length of the shortest vector in  $L^\perp$ , based on the Minkowski bound, to  $\Delta$ , in an average sense. The cross-over between  $L^\perp$  having short vectors or not is at  $S = 0$ .

The factor  $1/d$  in the definition of the function  $S$  is included to account for the orientations of  $L_0$  and of  $L^\perp$  with respect to the coordinate axes, and the averaging is done in such a way that it can be explicitly evaluated if  $d$  is much smaller than  $h$  but is not too small:

$$\begin{aligned} \mathbb{E}_h[\text{Vol}(L_{\text{approx}})^2] &= \left( \prod_{i=1}^n \frac{1}{i!} \right)^2 \times \\ &\quad \frac{1}{(2h+1)^d} \sum_{t_1, \dots, t_d = -h}^h \sum_{\substack{S \subseteq \{1, \dots, d\} \\ \#S = n+1}} \prod_{1 \leq k < \ell \leq n+1} (t_{S(k)} - t_{S(\ell)})^2 \\ &= \left( \prod_{i=1}^n \frac{1}{i!} \right)^2 \frac{\binom{d}{n+1}}{(2h+1)^{n+1}} \sum_{t_1, \dots, t_{n+1} = -h}^h \prod_{1 \leq k < \ell \leq n+1} (t_k - t_\ell)^2 \\ &\approx \left( \prod_{i=1}^n \frac{1}{i!} \right)^2 \binom{d}{n+1} (2h)^{n(n+1)} \\ &\quad \int \cdots \int_{[0,1]^{n+1}} \prod_{1 \leq k < \ell \leq n+1} (x_k - x_\ell)^2 dx_1 \cdots dx_{n+1}. \end{aligned}$$

The integral of the square of the Vandermonde determinant is equal to  $(n+1)!$  times the determinant of the  $(n+1) \times (n+1)$  Hilbert matrix  $H_{i,j} = 1/(i+j-1)$ , which has been calculated by Hilbert himself [12]

and is equal to

$$\det((H_{i,j})_{i,j=1}^{n+1}) = \prod_{i=1}^n (i!)^4 / \prod_{i=1}^{2n+1} i!.$$

We thus obtain

$$(17) \quad \begin{aligned} S(n, h, p, \Delta, d) &\approx \log\left(\frac{p}{d\Delta}\right) \\ &+ \frac{1}{2(d-n-1)} \left( \sum_{i=1}^n (\log((n+1+i)!) - \log(i!)) \right. \\ &\quad \left. - \log\left(\frac{d}{n+1}\right) - n(n+1) \log(2h) \right). \end{aligned}$$

**6.2. Tests.** We are mainly interested in parameter values corresponding to HIMMO, that is, when we have a polynomial of degree  $n$  and  $b$ -bit keys, we choose  $p$  to have precisely  $(n+2)b$  bits and let  $\Delta = \lfloor p/2^{b+1} \rfloor$ . We are interested only in reconstruction in intervals of length at most  $2^b$ . Note that in this case  $S$  becomes independent of  $p$ .

Figure 4 shows the reconstruction error  $(f(t) - \tilde{f}(t))/p$  versus  $x/2^b$  for a randomly chosen polynomial  $f$  of degree  $n = 5$ ,  $h = 2^{15}$ ,  $p$  a random 112-bit prime and  $d = 20$  and  $23$ , respectively. The  $d$  observation points are uniformly chosen in the interval  $[0, 2^{16})$ . For  $d = 20$ , the error appears to be random for all  $x$ , whereas for  $d = 23$  the error is close to 0 for small  $x$ .

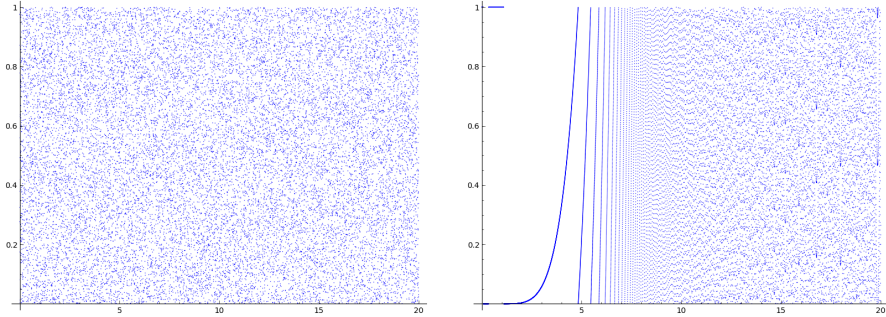


FIGURE 4. Reconstruction error  $(f(t) - \tilde{f}(t))/p$  versus  $x/2^b$  for  $n = 5$ ,  $b = 16$ ,  $p$  a random 112-bit prime and  $d = 20$  and  $23$ , respectively. The  $d$  observation points are uniformly chosen in the interval  $[0, 2^{16})$ .

Figure 5 shows graph of  $S(5, 2^{15}, p, \lfloor p/2^{17} \rfloor, d)$  as a function of  $d$ , showing that  $S$  becomes positive at  $d = 23$ . This suggests that an

approximate reconstruction is likely to be successful for  $d = 23$ , completely in line with our experimental results.

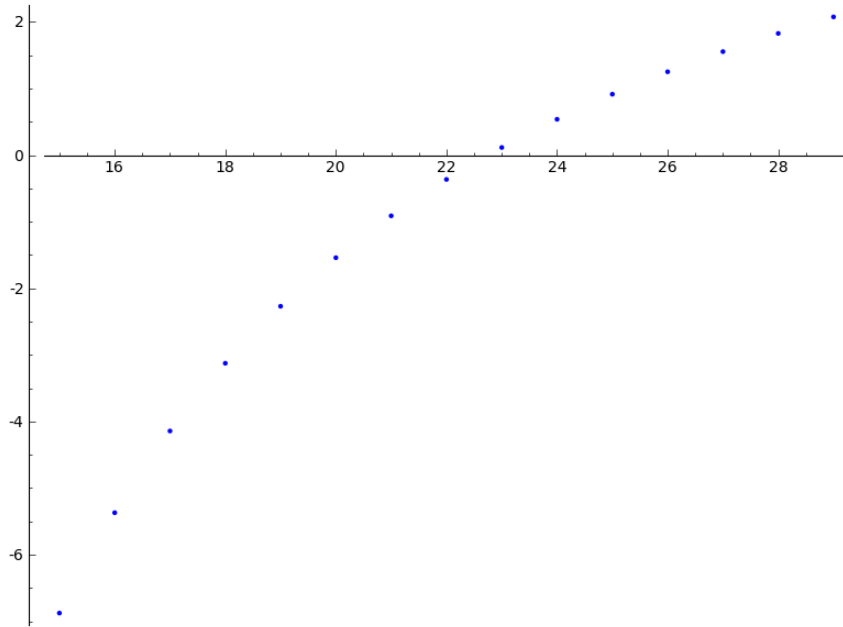


FIGURE 5. The function  $S(5, 2^{15}, p, \lfloor p/2^{17} \rfloor, d)$  as a function of  $d$  for large  $p$ .

For larger degree  $n = 26$ , and  $h = 2^{31}$ ,  $p$  a 896-bit prime,  $\Delta = \lfloor p/2^{33} \rfloor$ , the function  $S$  becomes positive for  $d = 426$ , which may be too large for a successful lattice attack on the present computers. But in a shorter interval,  $h = 128$ , the function  $S$  becomes positive at  $d = 73$ , see Figure 6.

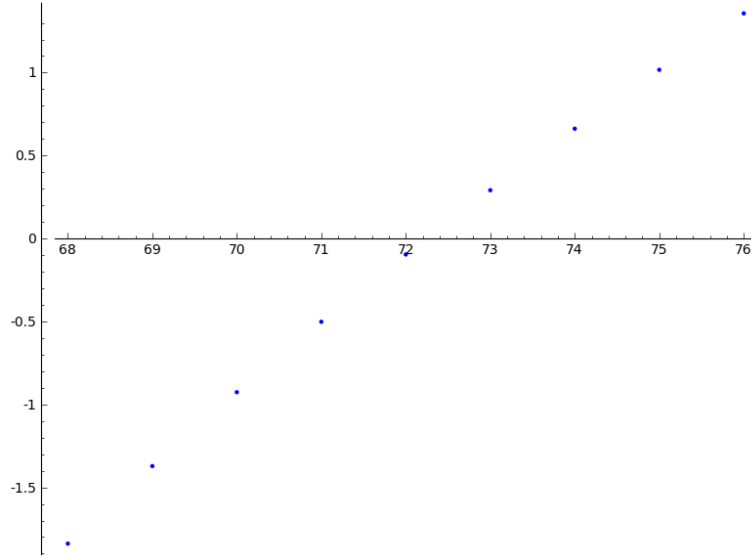


FIGURE 6. The function  $S(26, 2^{31}, p, \lfloor p/2^{33} \rfloor, d)$  as a function of  $d$  for large  $p$ .

The experimental results in Figures 7, 8 and 9 show how the reconstruction improves as  $d$  increases from 69 to 75. For  $c = 69$ , the reconstruction is close to zero only in the observation points (note that the points near the horizontal line  $y = 1$  are also considered to be close to zero). For  $c = 72$  also in a sizeable fraction of the not-observed points. Finally for  $c = 75$  in nearly all points of an interval of length 256.

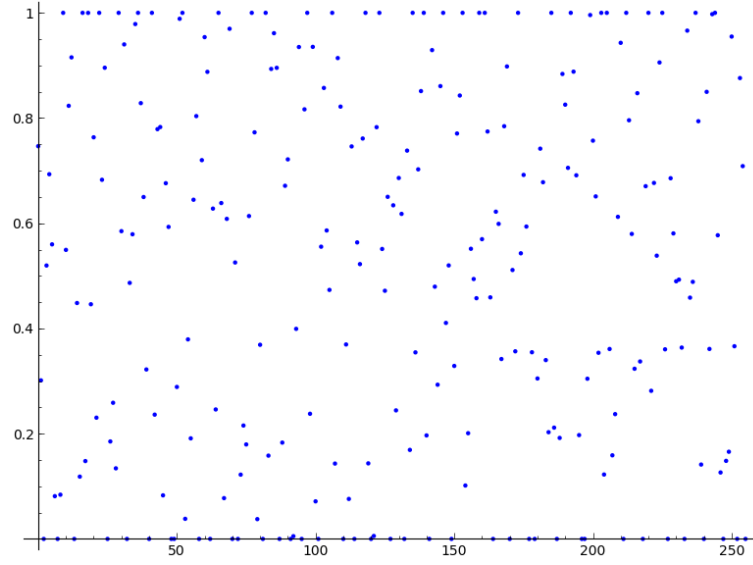


FIGURE 7. Reconstruction error for  $0 \leq x < 256$  with  $\alpha = 26$ ,  $b = 32$ ,  $w = 256$  for  $d = 69$ .

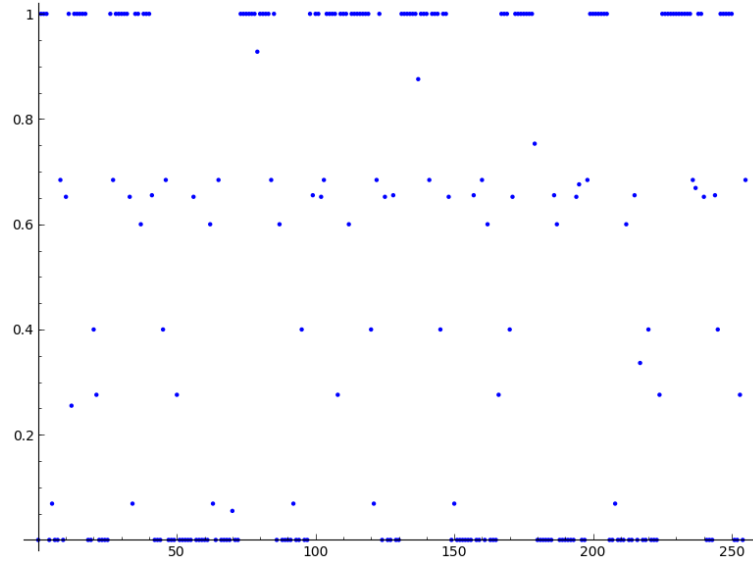


FIGURE 8. Reconstruction error for  $0 \leq x < 256$  with  $\alpha = 26$ ,  $b = 32$ ,  $w = 256$  for  $d = 72$ .

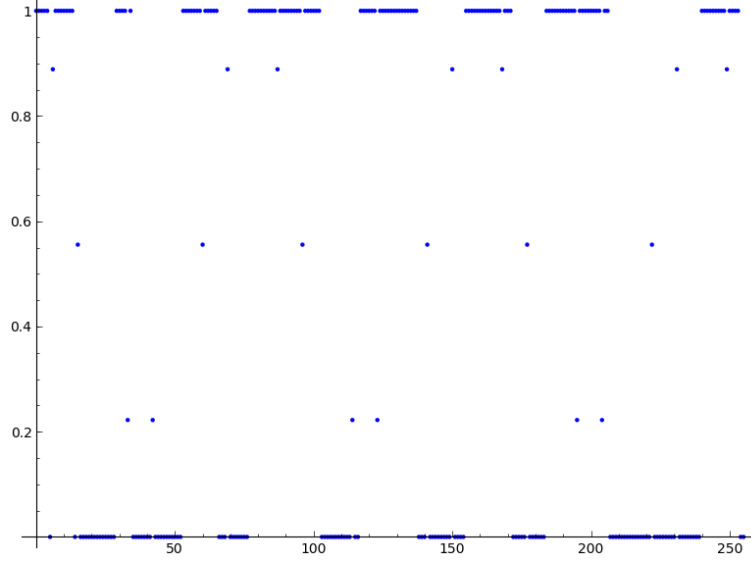


FIGURE 9. Reconstruction error for  $0 \leq x < 256$  with  $\alpha = 26$ ,  $b = 32$ ,  $w = 256$  for  $d = 75$ .

Finally, we note that for fairly small values of  $b$ , for example,  $b = 8$ ,  $S$  is negative for all  $d < 2^b$  if  $n$  is large enough, as shown for  $n = 10$  in Figure 10. This suggests that approximate reconstruction in the full interval  $[0, 255]$  cannot work, no matter how many observation points are used.

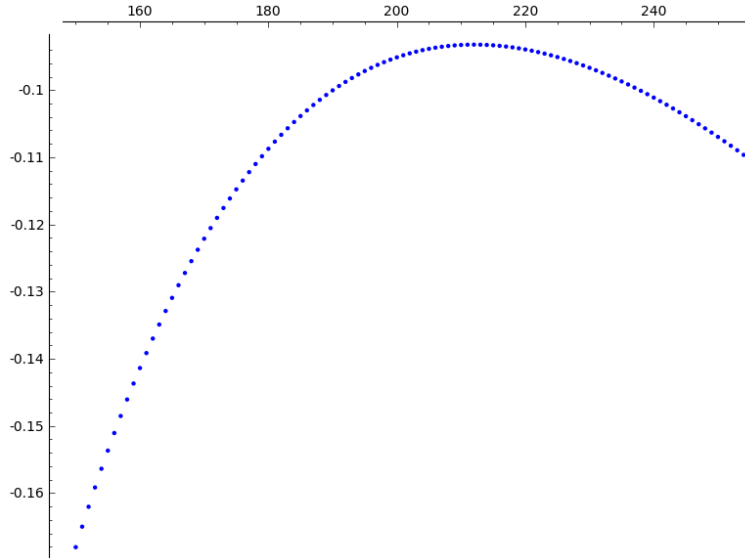


FIGURE 10.  $S(10, 2^7, p, \lfloor p/2^9 \rfloor, d)$  as a function of  $d$ .



## 7. COMMENTS

Clearly in practical implementations of our algorithm, using a certain approximate version of Lemma 7 is more natural. One can easily check that the algorithm and the result of Theorem 9 remain valid in this case as well.

In the settings of HIMMO with  $b$ -bit keys and  $b$ -bit identifiers we have  $h = 2^{b-1}$ ,  $2^{(n+2)b-1} < p < 2^{(n+2)b}$  and  $\Delta = \lfloor p/2^{b+1} \rfloor$ , see [8], so Theorem 9 does not apply. This can be considered as an indirect confirmation of the strength of HIMMO: recovery of the complete polynomial is impossible.

From studying (17) as a function of  $d$  for various values of  $b$ , we can obtain indications if an approximate recovery in an interval  $[-h, h]$  is likely to succeed, also for shorter intervals,  $h < 2^{b-1}$ , that do not contain all possible identifiers.

We also recall that there is a different approach to the hidden number problem, due to Akavia [1]. This approach does not use any lattice algorithms but rather examine the Fourier coefficients of the “hidden” linear function. However, the properties of these coefficients (large values near the origin and a smooth decay away from the origin) do not hold for non-linear polynomial functions, where all Fourier coefficients are expected to be of about the same size.

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